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NOTES AND COMMENTS

HETEROGENEOUS DEMAND AND ORDER OF RESOURCE EXTRACTION

BY UJJAYANT CHAKRAVORTY AND DARRELL L. KRULCE¹

1. INTRODUCTION

HERFINDAHL (1967) AND SOLOW AND WAN (1976) SHOW that several deposits of an exhaustible resource, identical except for different unit extraction costs, are optimally extracted in order of increasing extraction cost. Kemp and Long (1980) show that this need not be true in a general equilibrium framework, if the unit extraction cost of each deposit is constant in terms of a perfect substitute for the resource. Lewis (1982) proves that even in the Kemp-Long framework, least-cost deposits will be extracted first as long as the extracted resource can be converted into capital that can be consumed or stored for future consumption.

Lewis suggests that "in partial equilibrium models of resource production, it seems to be almost transparent that it is efficient to exploit low cost deposits first." We show that this need not be true if there is more than one demand for different resources or resource grades. If a low cost resource is relatively more efficient in supplying some demand, it will be optimal to extract a high and low cost resource simultaneously. This simultaneous extraction has the effect of conserving the low cost resource for the demand where it is relatively more efficient.

Standard models of resource extraction (including Solow-Wan and Kemp-Long) assume that quality differences between resource grades can be completely characterized by cost. We formulate a model of two resources which have a quality difference that cannot be characterized independently of demand. For illustrative purposes, we choose oil and coal as our two resources. We assume that there is no quality difference between these two resources when used to produce electricity but there is a significant quality difference if used for transportation. That is, either oil or coal can fire a power plant but it is easier to run an automobile on oil (gasoline) than on coal.

We characterize the optimal solution to this two-resource, two-demand problem and prove the existence of three distinct phases of resource extraction. In the first phase, the cheaper oil is extracted for both electricity and transportation. This is followed by a second phase in which oil is extracted for transportation but the more costly coal is extracted for electricity. In the final phase, oil is exhausted, and coal is extracted for both electricity and transportation. The main result relates to the second phase: both oil and coal are extracted simultaneously. During this phase, higher cost coal is extracted to generate electricity even though cheaper oil is available, in spite of there being no quality difference between the two resources in generating electricity. Intuitively, oil has a comparative advantage in transportation and so it is optimal to conserve oil for future transportation demand and generate electricity with coal, even though coal is more costly. The Lewis conjecture fails because oil is too valuable as a transportation fuel to be wasted generating electricity.

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Although we develop the model using quality differences between separate resources, the idea of demand-dependent quality differences applies to resource grades as well. For example, the quality difference between light and heavy crude oil matters less to an oil refiner with extensive cracking capability than one without. Thus there is a heterogeneous demand for different grades of crude. Similarly, the quality difference between low and high sulfur coal matters more in locations with strict air-quality standards.

Empirically, Adelman (1986) observes that resource deposits with widely different extraction costs are extracted simultaneously. By relaxing the standard assumption of homogeneous demand, we provide a possible explanation of this observation. This suggests that there is fertile ground for researching a general theory of resource extraction with multiple demands, where the assumption of homogeneous demand has been discarded.

2. A TWO RESOURCE, TWO DEMAND MODEL

Consider a simple model of two resource deposits (oil and coal, denoted O and C) and two demands for those resources (electricity and transportation, denoted E and T). We assume that both oil and coal can generate electricity equally well but that oil is relatively more efficient as a transportation fuel. We summarize this efficiency difference by allowing oil and coal to be used directly for electricity and oil to be used directly for transportation but requiring that coal be converted for use in transportation. The cost of converting a unit of coal for transportation is $z > 0$.²

Oil can be extracted at a unit cost of c_O and coal at a unit cost of c_C where $c_O < c_C$. Thus, oil is cheaper to extract than is coal. The initial stocks of oil and coal are $Q_O(t_0) > 0$ and $Q_C(t_0) > 0$ where the initial time is $t_0 = 0$. Fuel demand for electricity is a strictly positive, bounded, continuous, strictly decreasing function of price, $D_E(p)$, with $\int_0^\infty D_E(p) dp < \infty$.³ Similar assumptions hold for fuel demand for transportation, $D_T(p)$. We can imagine that these demands are derived from the final demand for electricity and transportation, taking into account possible substitutions of other energy sources.

The hypothetical planner chooses the quantity of resource extracted for each demand. We denote by $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$ the quantity of oil extracted for electricity, oil for transportation, coal for electricity, and coal for transportation over time. The problem is to choose the extraction functions that maximize the discounted social surplus. Given a discount rate $r > 0$, this can be posed as the following optimal control problem:

Choose $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$ to maximize

$$(1) \quad \int_{t_0}^{\infty} e^{-rt} \left[\int_0^{q_{OE}(t)+q_{CE}(t)} D_E^{-1}(x) dx + \int_0^{q_{OT}(t)+q_{CT}(t)} D_T^{-1}(x) dx - zq_{CT}(t) \right. \\ \left. - (q_{OE}(t) + q_{OT}(t))c_O - (q_{CE}(t) + q_{CT}(t))c_C \right] dt$$

subject to

$$(2) \quad q_{OE}(t) \geq 0, \quad q_{OT}(t) \geq 0, \\ q_{CE}(t) \geq 0, \quad q_{CT}(t) \geq 0, \quad Q_O(t) \geq 0, \quad Q_C(t) \geq 0,$$

$$(3) \quad \dot{Q}_O(t) = -q_{OE}(t) - q_{OT}(t), \quad \text{and} \quad \dot{Q}_C(t) = -q_{CE}(t) - q_{CT}(t).$$

² An apparently more general model would specify conversion costs from each resource to each demand. It is an easy matter, however, to normalize conversion costs by shifting the cost and demand functions. This normalization leaves a single relevant conversion cost.

³ This last restriction, used to guarantee a solution to the problem, implies a finite consumer surplus.

In the above, D_E^{-1} and D_T^{-1} are inverse demand functions. The first two terms in (1) denote the aggregate consumer benefit from electricity and transportation. The remaining terms in (1) are the conversion cost for coal to transportation and the total extraction cost for oil and coal. $Q_O(t)$ and $Q_C(t)$ are the remaining stocks of oil and coal over time, from which we obtain the familiar differential equations given in (3). The current value Hamiltonian for this problem is

$$\begin{aligned} H = & \int_0^{q_{OE}(t)+q_{CE}(t)} D_E^{-1}(x) dx + \int_0^{q_{OT}(t)+q_{CT}(t)} D_T^{-1}(x) dx - zq_{CT}(t) \\ & - (q_{OE}(t) + q_{OT}(t))c_O - (q_{CE}(t) + q_{CT}(t))c_C \\ & - (q_{OE}(t) + q_{OT}(t))\lambda_O(t) - (q_{CE}(t) + q_{CT}(t))\lambda_C(t) \end{aligned}$$

where $\lambda_O(t) \geq 0$ and $\lambda_C(t) \geq 0$ have the standard interpretation as scarcity rents.

Let the price of oil be $p_O(t) = c_O + \lambda_O(t)$, the price of coal be $p_C(t) = c_C + \lambda_C(t)$, the price of fuel for electricity be $p_E(t) \equiv D_E^{-1}(q_{OE}(t) + q_{CE}(t))$, and the price of fuel for transportation be $p_T(t) \equiv D_T^{-1}(q_{OT}(t) + q_{CT}(t))$. Then necessary conditions for a solution to program (1)–(3) are

$$(4) \quad \dot{Q}_O(t) = -q_{OE}(t) - q_{OT}(t),$$

$$(5) \quad \dot{Q}_C(t) = -q_{CE}(t) - q_{CT}(t),$$

$$(6) \quad \dot{\lambda}_O(t) = r\lambda_O(t),$$

$$(7) \quad \dot{\lambda}_C(t) = r\lambda_C(t),$$

$$(8) \quad p_E(t) \leq p_O(t), \quad \text{if } < \text{ then } q_{OE}(t) = 0,$$

$$(9) \quad p_E(t) \leq p_C(t), \quad \text{if } < \text{ then } q_{CE}(t) = 0,$$

$$(10) \quad p_T(t) \leq p_O(t), \quad \text{if } < \text{ then } q_{OT}(t) = 0,$$

$$(11) \quad p_T(t) \leq p_C(t) + z, \quad \text{if } < \text{ then } q_{CT}(t) = 0,$$

$$(12) \quad \lim_{t \rightarrow \infty} e^{-rt} \lambda_O(t) Q_O(t) = 0, \quad \text{and}$$

$$(13) \quad \lim_{t \rightarrow \infty} e^{-rt} \lambda_C(t) Q_C(t) = 0.$$

PROPOSITION 1: *There exists a unique optimal solution to program (1)–(3) and the necessary conditions (4)–(13) are also sufficient.*

PROOF: See Appendix.

Conditions (4)–(7) are standard in exhaustible resource problems. Conditions (8)–(11) say that the resource that has the lowest price plus conversion cost always supplies each demand. Conditions (12) and (13) are the transversality conditions. As part of the optimal control solution, q_{OE} , q_{OT} , q_{CE} , and q_{CT} are piecewise continuous and λ_O and λ_C are continuous.

3. CHARACTERIZATION OF THE OPTIMAL SOLUTION

The optimal trajectories for the extraction of oil and coal are completely characterized by the following:

PROPOSITION 2: (a) *There exist unique switchpoints t_R (where coal replaces oil for electricity) and t_S (where coal replaces oil for transportation) such that $t_0 \leq t_R < t_S < \infty$ and*

$$\begin{aligned} q_{OE}(t) > 0 \quad \text{and} \quad q_{CE}(t) &= 0 \quad \text{for } t \in (t_0, t_R), \\ q_{OE}(t) &= 0 \quad \text{and} \quad q_{CE}(t) > 0 \quad \text{for } t \in (t_R, \infty), \\ q_{OT}(t) > 0 \quad \text{and} \quad q_{CT}(t) &= 0 \quad \text{for } t \in (t_0, t_S), \quad \text{and} \\ q_{OT}(t) &= 0 \quad \text{and} \quad q_{CT}(t) > 0 \quad \text{for } t \in (t_S, \infty). \end{aligned}$$

(b) *If $\int_{c_C}^{c_C+z} D_T(p)/(p - c_O) dp < rQ_O(t_0)$, then further, $t_0 < t_R$.*

PROOF: See Appendix.

Proposition 2 says that there are three distinct phases of resource extraction as shown in Figure 1. In the first phase, until time t_R , the price of oil (extraction cost plus scarcity rent) is cheaper than that of coal and so only oil is extracted for both electricity and transportation. The cheaper oil commands a higher scarcity rent and so the price of oil rises faster than the price of coal. Beyond time t_R , the price of oil exceeds the price of coal, but not enough to offset the cost of converting coal to transportation. Thus in the second stage, coal is extracted for electricity and oil is extracted for transportation. Finally, at time t_S , the price of oil is high enough that coal is extracted for both electricity and transportation.⁴ Since oil is not extracted beyond time t_S , in view of the Lemma (see Appendix), oil is exhausted at time t_S .

Although the second and third phases must exist, it is possible that the first phase does not. Part (b) of Proposition 2 provides a sufficient condition for the first phase (where only oil is extracted for both electricity and transportation) to exist. Note that $\int_{c_C}^{c_C+z} D_T(p)/(p - c_O) dp$ decreases with c_C and increases with c_O , z , and $D_T(p)$. Thus it may be optimal to extract oil for both electricity and transportation at first if the cost of oil is low (small c_O), the cost of the alternative fuel coal is high (large c_C), or oil is plentiful (large $Q_O(t_0)$). Conversely, it may be optimal to save oil for future consumption if its comparative advantage is high (large z), the demand for oil's more efficient use is large (high $D_T(p)$), or the discount rate is low (small r).

Of particular interest is the simultaneous extraction of coal and oil during the second phase. Coal rather than oil is extracted for electricity even though oil is strictly cheaper and both resources have the same efficiency for electricity. This is in opposition to conventional wisdom that if efficiencies are equal, only the cheaper resource is extracted until it is exhausted. Here, it is optimal to conserve oil for its relatively more efficient use in transportation. This result is completely dependent on a nonzero z , the comparative advantage of oil for transportation. If $z = 0$, the second phase vanishes and oil is completely exhausted before the exploitation of coal begins.

Since simultaneous extraction results from a comparative advantage between resources, it is natural to ask if there can be a case of complete specialization, where each resource always exclusively supplies its relatively most efficient use. In our model, complete specialization cannot occur since $t_S < \infty$. Beyond time t_S , when oil is exhausted,

⁴ Note that the cost functions for electricity and transportation are discontinuous. The fuel cost for a unit of electricity, given by $(c_O q_{OE}(t) + c_C q_{CE}(t))/D_E(p_E(t))$, jumps upward by an amount $c_C - c_O$ at time $t = t_R$. Similarly, the fuel cost for a unit of transportation jumps by $c_C - c_O + z$ at time $t = t_S$.

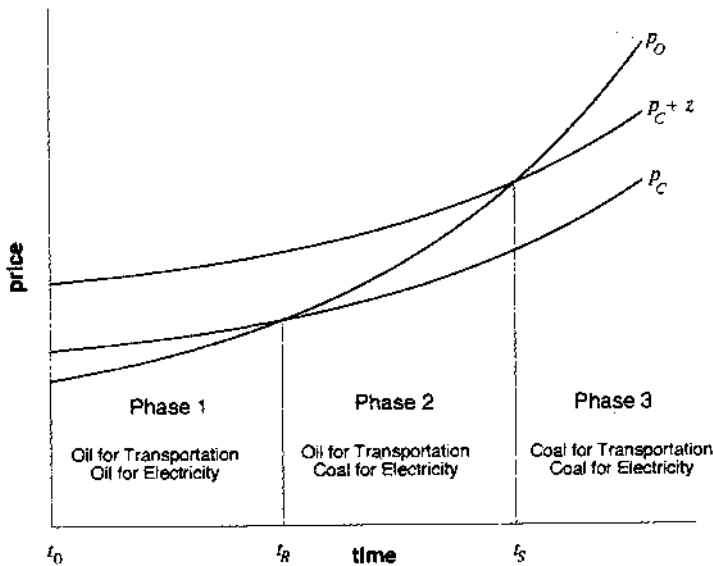


FIGURE 1.—Phases of resource extraction.

coal always supplies both demands. It is because oil is the cheaper resource that it is exhausted first.

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APPENDIX

PROOF OF PROPOSITION 1

We follow Farzin (1992) in using Seierstad and Sydsæter (1987, Theorem 15, p. 237) to prove the existence of a unique optimal solution to (1)–(3) and Theorem 13 (p. 234) to prove that the necessary conditions are sufficient. Defining $f_0(q_{OE}, q_{OT}, q_{CE}, q_{CT}, t)$ to be the integrand of (1), $f_1(q_{OE}, q_{OT}) \equiv -q_{OE} - q_{OT}$, and $f_2(q_{CE}, q_{CT}) \equiv -q_{CE} - q_{CT}$, these theorems reduce to proving the following six conditions.

CONDITION 1: $f_0(q_{OE}, q_{OT}, q_{CE}, q_{CT}, t)$, $f_1(q_{OE}, q_{OT})$ and $f_2(q_{CE}, q_{CT})$ are continuous.

PROOF: Inspection of the functions reveals that they are all continuous.

CONDITION 2: $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$ are bounded.

PROOF: Since $D_E(p)$ is bounded, then $q_{OE}(t)$ and $q_{CE}(t)$ are bounded. Similarly, since $D_T(p)$ is bounded, then $q_{OT}(t)$ and $q_{CT}(t)$ are bounded.

CONDITION 3: For any admissible $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$, there exists a piecewise continuous function $\phi_0(t)$ with $\int_{t_0}^{\infty} \phi_0(t) dt < \infty$ such that $f_0(q_{OE}(t), q_{OT}(t), q_{CE}(t), q_{CT}(t), t) \leq \phi_0(t)$.

PROOF: Since $\int_{t_0}^{\infty} D_E(p) dp < \infty$ and $\int_{t_0}^{\infty} D_T(p) dp < \infty$, then $\int_{t_0}^{q_{OE}(t)+q_{CE}(t)} D_E^{-1}(x) dx$ and $\int_{t_0}^{q_{OT}(t)+q_{CT}(t)} D_T^{-1}(x) dx$ are bounded. Since all of the other terms in the square brackets of (1) are also bounded, then $e^{rt} f_0(q_{OE}(t), q_{OT}(t), q_{CE}(t), q_{CT}(t), t)$ is bounded, say by K . Let $\phi_0 = Ke^{-rt}$ so that $\int_{t_0}^{\infty} \phi_0(t) dt = K/r < \infty$. Then

$$f_0(q_{CE}(t), q_{CT}(t), q_{OT}(t), q_{OE}(t), t) \leq Ke^{-rt} = \phi_0(t).$$

CONDITION 4: For any admissible $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$, there exists piecewise continuous functions $\phi_1(t)$ and $\phi_2(t)$ with $\int_{t_0}^{\infty} \phi_1(t) dt < \infty$ and $\int_{t_0}^{\infty} \phi_2(t) dt < \infty$ such that $|f_1(q_{OE}(t), q_{OT}(t))| \leq \phi_1(t)$ and $|f_2(q_{CE}(t), q_{CT}(t))| \leq \phi_2(t)$.

PROOF: Let $\phi_1(t) = -\dot{Q}_O(t)$. Then

$$\int_{t_0}^{\infty} \phi_1(t) dt = -\int_{t_0}^{\infty} \dot{Q}_O(t) dt = -\lim_{t \rightarrow \infty} Q_O(t) + Q_O(t_0) \leq Q_O(t_0)$$

where the last inequality follows from (2), $Q_O(t) \geq 0$. Since from (2), $q_{OE}(t) \geq 0$ and $q_{OT}(t) \geq 0$ which from (3) implies that $\dot{Q}_O(t) \leq 0$, then

$$|f_1(q_{OE}(t), q_{OT}(t))| = |-q_{OE}(t) - q_{OT}(t)| = |\dot{Q}_O(t)| = -\dot{Q}_O(t) = \phi_1(t).$$

The existence of $\phi_2(t)$ follows symmetrically.

CONDITION 5: For any admissible $q_{OE}(t)$, $q_{OT}(t)$, $q_{CE}(t)$, and $q_{CT}(t)$, there exists nonnegative functions $a(t)$ and $b(t)$ such that $\|f_1(q_{OE}(t), q_{OT}(t)), f_2(q_{CE}(t), q_{CT}(t))\| \leq a(t)\|Q_O(t), Q_C(t)\| + b(t)$.

PROOF: Since $f_1(q_{OE}(t), q_{OT}(t)) = -q_{OE}(t) - q_{OT}(t)$, from Condition 2, $f_1(q_{OE}(t), q_{OT}(t))$ is bounded. Similarly, $f_2(q_{CE}(t), q_{CT}(t))$ is bounded and so $\|f_1(q_{OE}(t), q_{OT}(t)), f_2(q_{CE}(t), q_{CT}(t))\|$ is bounded. Let $b(t)$ be this bound and let $a(t) = 0$ to obtain the result.

CONDITION 6: The function $f_0(q_{OE}, q_{OT}, q_{CE}, q_{CT}, t)$ is concave for all t .

PROOF: Since $D_E'(p) < 0$ and $D_T'(p) < 0$, then $f_0(q_{OE}, q_{OT}, q_{CE}, q_{CT}, t)$ is concave for all t . Q.E.D.

PROOF OF PROPOSITION 2

Solving (6) and (7) produce the Hotelling equations,

$$(14) \quad \lambda_O(t) = \lambda_O(t_0)e^{rt} \quad \text{and}$$

$$(15) \quad \lambda_C(t) = \lambda_C(t_0)e^{rt}$$

which state that scarcity rents rise at the rate of interest. Because demand is positive at all prices, both oil and coal will eventually be economically valuable and will be extracted. We show with the following lemma that both resources will thus approach exhaustion.

LEMMA: $\lim_{t \rightarrow \infty} Q_O(t) = 0$ and $\lim_{t \rightarrow \infty} Q_C(t) = 0$.

PROOF: Suppose that $\lambda_O(t_0) = 0$. Then from (14), $\lambda_O(t) = 0$, hence $p_O(t) = c_O$ and so from (8), $p_E(t) \leq c_O$. Since demand is positive and downward sloping,

$$0 < D_E(c_O) \leq D_E(p_E(t)) = q_{OE}(t) + q_{CE}(t).$$

Thus $\int_{t_0}^{\infty} (q_{OE}(t) + q_{CE}(t)) dt = \infty$ and so either $\int_{t_0}^{\infty} q_{OE}(t) dt = \infty$ or $\int_{t_0}^{\infty} q_{CE}(t) dt = \infty$. From (4), $Q_O(t) = -q_{OE}(t) - q_{OT}(t) \leq -q_{OE}(t)$, so if $\int_{t_0}^{\infty} q_{OE}(t) dt = \infty$, then eventually $Q_O(t)$ will become

negative which contradicts (2). Similarly, if $\int_{t_0}^{\infty} q_{CE}(t) dt = \infty$, then $Q_C(t)$ will become negative which again contradicts (2). Thus the supposition is false and so $\lambda_O(t_0) > 0$. Combining (12) and (14) yields

$$0 = \lim_{t \rightarrow \infty} e^{-rt} \lambda_O(t) Q_O(t) = \lim_{t \rightarrow \infty} e^{-rt} \lambda_O(t_0) e^{rt} Q_O(t) = \lambda_O(t_0) \lim_{t \rightarrow \infty} Q_O(t)$$

which since $\lambda_O(t_0) > 0$ implies that $\lim_{t \rightarrow \infty} Q_O(t) = 0$. By symmetry, $\lim_{t \rightarrow \infty} Q_C(t) = 0$.

Using the Lemma, we now prove Proposition 2. *Part (a).* Define $\varphi(t) \equiv p_O(t) - p_C(t)$. If $\varphi(t) < 0$, then $p_O(t) < p_C(t)$. Combining with (8) yields $p_E(t) < p_C(t)$ which from (9) implies that $q_{CE}(t) = 0$. Then

$$0 < D_E(p_E(t)) = q_{OE}(t) + q_{CE}(t) = q_{OE}(t).$$

That is,

$$(16) \quad \varphi(t) < 0 \Rightarrow q_{CE}(t) = 0 \Rightarrow q_{OE}(t) > 0.$$

Similarly,

$$(17) \quad \varphi(t) > 0 \Rightarrow q_{OE}(t) = 0 \Rightarrow q_{CE}(t) > 0$$

and from (10) and (11),

$$(18) \quad \varphi(t) < z \Rightarrow q_{CT}(t) = 0 \Rightarrow q_{OT}(t) > 0 \quad \text{and}$$

$$(19) \quad \varphi(t) > z \Rightarrow q_{OT}(t) = 0 \Rightarrow q_{CT}(t) > 0.$$

Combining (16), (18), and (5) yields

$$(20) \quad \varphi(t) < 0 \Rightarrow \dot{Q}_C(t) = -q_{CE}(t) - q_{CT}(t) = 0$$

and similarly, from (17), (19), and (4),

$$(21) \quad \varphi(t) > z \Rightarrow \dot{Q}_O(t) = 0.$$

We complete the proof by showing that $\varphi(t)$ is continuous, strictly increasing, and unbounded and that $\varphi(t_0) < z$. Then from (16)–(19), t_S is the unique point where $\varphi(t_S) = z$ and t_R is the unique point where $\varphi(t_R) = 0$ unless $\varphi(t_0) > 0$ in which case $t_R = t_0$.

From the definition of $p_O(t)$ and $p_C(t)$, (14) and (15),

$$(22) \quad \begin{aligned} \varphi(t) &= (c_O + \lambda_O(t_0)e^{rt}) - (c_C + \lambda_C(t_0)e^{rt}) \\ &= (c_O - c_C) + (\lambda_O(t_0) - \lambda_C(t_0))e^{rt} \end{aligned}$$

which is continuous since $\lambda_O(t)$ and $\lambda_C(t)$ are continuous. Suppose that $\lambda_O(t_0) \leq \lambda_C(t_0)$ and hence that $\varphi(t)$ is nonincreasing. Then since $c_O < c_C$, (22) implies that $\varphi(t) < 0$. Then from (20), $\dot{Q}_C(t) = 0$. That is, coal is never extracted. Since this contradicts the Lemma, the supposition is false and thus $\lambda_O(t_0) > \lambda_C(t_0)$. Then from (22), $\varphi(t)$ is strictly increasing and unbounded.

Suppose that $\varphi(t_0) \geq z$. Then since $\varphi(t)$ is strictly increasing, $\varphi(t) > z$ for $t \in (t_0, \infty)$. Then from (21), $\dot{Q}_O(t) = 0$. That is, oil is never extracted. Since this contradicts the Lemma, the supposition is false and so $\varphi(t_0) < z$.

Part (b). Consider the case that $\int_{c_C}^{c_C+z} D_T(p)/(p - c_O) dp < rQ_O(t_0)$ and suppose that $t_0 = t_R$. Then from Part (a),

$$(23) \quad q_{OE}(t) = 0 \quad \text{for} \quad t \in (t_0, \infty).$$

Since $t_0 = t_R$ implies that $\varphi(t_0) \geq 0$, then from (22), $\lambda_O(t_0) - \lambda_C(t_0) \geq c_C - c_O$ and so

$$(24) \quad \varphi(t) \geq (c_O - c_C) + (c_C - c_O)e^{rt}.$$

Let $t_N = \log(1 + z/(c_C - c_O))/r$. Combining with (24) yields $\varphi(t) > z$ for $t \in (t_N, \infty)$ which from (19) implies that

$$(25) \quad q_{OT}(t) = 0 \quad \text{for} \quad t \in (t_N, \infty).$$

Let $\gamma(t) = c_O + (c_C - c_O)e^{rt}$ so that $\gamma(t_0) = c_C$ and $\gamma(t_N) = c_C + z$. Then from the definition of

$\varphi(t)$ and (24), and since $p_C(t) \geq c_C$,

$$(26) \quad p_O(t) = \varphi(t) + p_C(t) \geq \varphi(t) + c_C \geq c_O + (c_C - c_O)e^{rt} = \gamma(t).$$

From (10),

$$p_T(t) = p_O(t) \Rightarrow D_T(p_O(t)) = D_T(p_T(t)) \geq q_{OT}(t) \quad \text{and}$$

$$p_T(t) < p_O(t) \Rightarrow q_{OT}(t) = 0 < D_T(p_O(t))$$

and therefore

$$(27) \quad D_T(p_O(t)) \geq q_{OT}(t).$$

Thus,

$$(28) \quad \int_{c_C}^{c_C+z} \frac{D_T(p)}{p - c_O} dp = \int_{\gamma(t_0)}^{\gamma(t_N)} \frac{D_T(p)}{p - c_O} dp \\ = \int_{t_0}^{t_N} \frac{D_T(\gamma(t))}{\gamma(t) - c_O} \gamma'(t) dt = r \int_{t_0}^{t_N} D_T(\gamma(t)) dt \\ \geq r \int_{t_0}^{t_N} D_T(p_O(t)) dt \geq r \int_{t_0}^{\infty} q_{OT}(t) dt = rQ_O(t_0)$$

where the third equality follows from $\gamma'(t) = r(\gamma(t) - c_O)$, the first inequality follows from (26) and that demand is downward sloping, the second inequality follows from (25) and (27), and the last equality follows from (23) and the Lemma. Since (28) contradicts the premise, the supposition is false and thus $t_0 < t_R$. Q.E.D.

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